USE OF TRANSFER MATRICES IN THE SEISMIC ANALYSIS OF TALL BUILDINGS

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SYNOPSIS

This paper present the use of transfer matrices in the seismic analysis of tall buildings. It is assumed in this work that most tall building structures can fit a mathematical model consisting of both a pure shear and a pure bending beams fixed at their bases and continuously connected through the height. The overall stiffness parameters characterizing the model are generated by addition of the individual values of each building substructure. A linear differential equation is formed and solved using the transfer matrix discretization technique.

The method gives fairly accurate results compared to standard matrix analysis solutions and provides a fast tool for preliminary design.

For the case of regular buildings, a set of graphs defining the main results of the analysis as a function of the stiffness parameters are given, and their use could further reduce the computational efforts.

Finally two examples show some of the applications of the method.

MATHEMATICAL MODEL

Figure 1 shows a structural model consisting of a shear beam and a slender beam fixed at the base and continuously connected through the height H, such that for any distributed lateral load $p(x)$, both beams have the same deflected shape.

The model is characterized by two stiffness parameters, $C_1(x)$ and $C_2(x)$ given by :

$$C_1(x) = \frac{GA}{\kappa}(x) \quad \text{and} \quad C_2(x) = EI(x) \qquad \ldots(1)$$

where :

G = shear modulus of elasticity of the shear beam
A = cross sectional area of the shear beam
κ = shear shape factor of the shear beam
E = modulus of elasticity of the slender beam
I = cross sectional moment of inertia of the slender beam

Denoting by $p_1(x)$ and $p_2(x)$ the lateral loads per unit length acting on the shear and the slender beam respectively, the following expressions can be written :

$$p_1(x) = -[C_1(x)\,y'(x)]'$$
$$p_2(x) = [C_2(x)\,y''(x)]'' \qquad \ldots(2)$$

where primes denote differentiation with respect to x.

From equilibrium

$$[C_2(x)\,y''(x)]'' - [C_1(x)\,y'(x)]' = p(x) \qquad \ldots(3)$$

Equation (3) is the governing differential equation for this model under any distributed lateral load.

This equation can be easily extended to inertia forces and becomes :

$$\frac{\partial^2}{\partial x^2}\left[C_2(x)\frac{\partial^2 y}{\partial x^2}\right] - \frac{\partial}{\partial x}\left[C_1(x)\frac{\partial y}{\partial x}\right] = -\mu(x)\frac{\partial^2 y}{\partial t^2} \qquad \ldots(4)$$

where $\mu(x)$ is the mass per unit length.

Using separation of variables such that :

$$y(x, t) = \phi(x)\,T(t) \qquad \ldots(5)$$

the modal equation (6) is obtained

$$[C_2(x)\,\phi''(x)]'' - [C_1(x)\,\phi'(x)]' = \omega^2\mu(x)\,\phi(x) \qquad \ldots(6)$$

ω is the parameter that represents the natural frequencies of the system and $\phi(x)$ represents the normal modes of vibration. The boundary conditions are :

$$|\phi(x)|_{x=0} = 0 \; ; \; |\phi'(x)|_{x=0} = 0 \qquad \ldots(7)$$
$$|\phi''(x)|_{x=H} = 0 ; \; |\{C_2(x)\phi''(x)\}' - \{C_1(x)\phi'(x)\}|_{x=H} = 0$$

Analytical solutions of equation (6) are possible in a very limited number of cases so that numerical techniques should be generally used.

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In this paper, the transfer matrix method is used, defining a state vector $\{S_i\}$ at a position x_i by

$$\{S_i\}=\begin{Bmatrix}\phi\ (x_i)\\ \phi'\ (x_i)\\ \phi''(x_i)\\ Q\ (x_i)\end{Bmatrix} \qquad ...(8)$$

where $Q(x_i)$ represents the total shear force at position x_i.

From the state vector at x_i it is possible to obtain an approximation of the state vector at a neighbour position x_{i+1} using Taylor series and equilibrium equations. Such relation becomes

$$\begin{Bmatrix}\phi\,(x_{i+1})\\ \phi'\,(x_{i+1})\\ \phi''\,(x_{i+1})\\ Q(x_{i+1})\end{Bmatrix}=\begin{bmatrix}1 & \left(l+\frac{C_1l^3}{6C_2}\right) & \left(\frac{l^2}{2}-\frac{C_2'l^3}{6C_2}\right) & -\frac{l^3}{6C_2}\\ 0 & \left(1+\frac{C_1l^2}{2C_2}\right) & \left(l-\frac{C_2'l^2}{2C_2}\right) & -\frac{l^2}{2C_2}\\ 0 & \frac{C_1l}{C_2} & \left(1-\frac{C_2'l}{C_2}\right) & -\frac{l}{C_2}\\ -\omega^2\mu l & -\frac{\omega^2\mu l^2}{2} & -\frac{\omega^2\mu l^3}{4} & 1\end{bmatrix}\begin{Bmatrix}\phi\,(x_i)\\ \phi'\,(x_i)\\ \phi''\,(x_i)\\ Q\,(x_i)\end{Bmatrix} \qquad ...(9)$$

l is the length of the interval i, $i+1$ $(l=x_{i+1}-x_i)$ and μ, C_1, C_2 and C_2' are the average values of $\mu(x)$, $C_1(x)$, $C_2(x)$ and $C_2'(x)$ in the interval.

$$\{S_{i+1}\}=[T_{i+1,i}]\,\{S_i\} \qquad ...(10)$$

where $[T_{i+1,\,i}]$ is defined as the transfer matrix from station i to station $i+1$.

Equation (10) is used from $i=0$ to $i=n$ where $x_n=H$ and the transfer matrix $[T_{n,0}]$ from base to top of the model is given by

$$[T_{n,0}]=[T_{n,n-1}]\,[T_{n-1,\,n-2}]\cdots[T_{1,0}] \qquad ...(11)$$

Explicitly:

$$\begin{Bmatrix}\phi\ (H)\\ \phi'\,(H)\\ \phi''(H)\\ Q\,(H)\end{Bmatrix}=\begin{bmatrix}T_{11} & T_{12} & T_{13} & T_{14}\\ T_{21} & T_{22} & T_{23} & T_{24}\\ T_{31} & T_{32} & T_{33} & T_{34}\\ T_{41} & T_{42} & T_{43} & T_{44}\end{bmatrix}\begin{Bmatrix}\phi\ (0)\\ \phi'\,(0)\\ \phi''(0)\\ Q\,(0)\end{Bmatrix} \qquad ...(12)$$

Using the boundary conditions $0=\phi(0)=\phi'(0)=\phi''(H)=Q(H)$, nontrivial solutions correspond to the condition

$$\triangle=T_{33}\,T_{44}-T_{34}\,T_{43}=0 \qquad ...(13)$$

which is satisfied by n different values of ω. In structural analysis it is important to know the lowest frequencies, generally the first three to five. The transfer matrix technique gives good accuracy with a small number of intervals.

REPRESENTATION OF ACTUAL STRUCTURES

An actual structure is a set of substructures such as walls, frames, etc. It is assumed (2) that substructure j has stiffness parameters C_{1j} and C_{2j} and satisfies the equation :

$$[C_{2j}(x)\,y''_j(x)]''-[C_{1j}(x)\,y'_j(x)]'=P_j(x) \qquad ...(14)$$

If $y(x)$ denotes the deflected shape of the complete structure under a distributed lateral load $p(x)$, geometric compatibility and equilibrium lead to :

$$[C_2(x)\,y''(x)]''-[C_1(x)\,y'(x)]'=p(x) \qquad ...(15)$$

which coincides with the governing differential equation (3).

In equation (15)

$$C_2(x)=\sum_{j=1}^{m} C_{2j}(x)$$

$$C_1(x)=\sum_{j=1}^{m} C_{1j}(x), \text{ and}$$

m = number of substructures ...(16)

It can be noted that the problem has been reduced to the determination of the individual stiffness parameters of the substructure types that are present in actual structures.

Some examples of actual substructure types are:

—Slender walls : $C_{1j}=0$; $C_{2j}=(EI)_j$

—Frame with relatively stiff beams (axial deformations not included)

$$C_{1j}=h\sum_i (a\,K_c)_i\ ;\ C_{2j}=0$$

where h is the story height, $(a\,K_c)_i$ is the reduced shear stiffness of column i of the j frame computed by Muto's method (3).

For prismatic columns, $\frac{1}{K_c}=\frac{\kappa h}{GA}+\frac{h^3}{12EI}$.

—Frame with relatively slender beams (axial deformations not included)

$$C_{1j}=\frac{1}{h}\sum_i (a K_b)_i L_i^2\ ;\ C_{2j}=\sum_i (EI)_i$$

where h is the story height, $(a\, K_b)_i$ is the reduced stiffness of the beam at span i, L_i is the length of the span i measured between column axis.

For prismatic beams, $\frac{1}{K_b}=\frac{\kappa L_1}{GA}+\frac{L_1^3}{12EI}$

where L_1 is the free length of the i^{th} span. Parameter a is computed by Muto's method interchanging beams and columns. If axial deformations have to be included, as it is the case of two walls connected by beams, C_{1j} and C_{2j} change to incorporate some contribution of the area in the moment of inertia of the cross section (1).

ANALYSIS OF BUILDINGS WITHOUT SETBACKS

Most of tall building structures are regular in plan and composed of frames of constant properties through the whole height of the buildings, and walls with constant length and constant or linearly decreasing thicknesses. Figures 2 to 16 may be used in order to obtain a fairly good estimation of the response of such structures under seismic loading.

C_1 and μ are supposed to be constant. The variation of wall thicknesses is taken into account by means of the following parameters :

$$C_2=\frac{C_2\,(x=0)+C_2\,(x=H)}{2}$$

$$\beta=\frac{C_2\,(x=0)-C_2\,(x=H)}{C_2\,(x=0)}\cdot 100\,[\%] \qquad \ldots(17)$$

$$\alpha^2=\frac{C_1\,H^2}{C_2}$$

The graphs have been prepared for $\beta=25\%$ and they can be used for β from 0 to 50% (usual range of variation) with an error less than 5%.

First three natural frequencies are obtained by the formula

$$\omega_i=\delta_i\sqrt{\frac{C_2}{\mu H^4}} \qquad \ldots(18)$$

The frequency coefficient δ_i is obtained from figure 2. i denotes mode number.

The equivalent masses for the computation of the base shear for the first three modes are obtained by

$$M_i^*=m_i^*\,\mu H, \qquad \ldots(19)$$

m_i^* is given in figure 3.

Maximum base shear for the first three modes is computed by

$$Q_{oi}=M_i^*\,S_{\bar{A}i} \qquad \ldots(20)$$

where $S_{\bar{A}i}$ is the value of pseudo accelerations spectrum corresponding to natural frequency ω_i of mode i.

The deflected shape :

$$y_i(x)=\frac{Q_{oi}H^3}{C_2\delta_i^2F_i}\,\bar{y}_i(x) \qquad \ldots(21)$$

where $F_i=\frac{1}{H}\int_0^H \bar{y}_i(x)\,dx$ is the scaling factor for mode i, and its first derivative

$$y_i'(x)=\frac{Q_{oi}H^2}{C_2}\,\bar{y}_i'(x) \qquad \ldots(22)$$

are computed for the first three modes with the values of F, figure 4, and the values of $\bar{y}(x)$ and $\bar{y}'(x)$ from figures 5, 6; 9, 10; 13, 14.

The deflected shape is used for the design of separations between buildings or parts of a building; its first derivate is used for checking the relative inter-story displacement.

Shear and overturning moment diagrams for the whole building structure are obtained for the first mode by

$$\left.\begin{aligned} Q_i(x)&=Q_{oi}\,\bar{Q}_i(x)\\ M_i(x)&=Q_{oi}H\bar{M}_i(x)\end{aligned}\right\} \qquad \ldots(23)$$

where $\bar{Q}_i(x)$ and $\bar{M}_i\,(x)$ are given in figures 7, 8 ; 11, 12; 15, 16.

The method also gives the shear forces $Q_1(x)=-C_1(x)\,y'(x)$ and $Q_2(x)=Q(x)-Q_1(x)$ carried for each mode by the shear beam and the slender beam of the model; then it is easy to compute the shear diagram of each frame and wall, by just distributing the partial shear diagrams in proportion to the C_{1j} or C_{2j} respectively ($j^{th.}$ substructure). Properties for the first three modes are then superposed according to design code provisions.

EXAMPLES

In order to show the applications of the technique described in this paper, two examples are presented. Metric tons, meters and seconds are used.

The first example (figure 17 a.) is a 20-story shear wall-frame building. Geometrical and mechanical properties are :

$$E = 3.500.000\ [t/m^2]$$
$$G = 1.400.000\ [t/m^2]$$

Story	Beams width [m] / depth [m]	Columns width [m] / depth [m]	Wall thickness [m]	Weight [t]	Height [m]
20	0.70/1.00	0.70/0.70	0.20	200	2.80
19—16	do	0.70/0.70	0.20	400	do
15—11	do	0.80/0.80	0.25	400	do
10— 6	do	0.90/0.90	0.25	400	do
5— 1	do	1.00/1.00	0.30	400	do

Stories are indexed starting from the bottom.

Next table shows some comparative results using: (a) Discrete matrix solution, neglecting axial and shear deformations in the elements;

(b) transfer matrix method and Muto's approach to determine $C_1(x)$ and $C_2(x)$:

(c) graphical solution using figures 1 and 2 and mean values for C_1 and C_2.

Vibration parameter	Case (a)	Case (b)	Case (c)
1st. period [sec]	1.117	0.882	0.867
2nd. period [sec]	0.252	0.223	0.216
3rd. period [sec]	0.097	0.092	0.090
1st. equivalent mass [%]	63.9	65.5	66.8
2nd. equivalent mass [%]	17.9	13.7	12.8
3rd. equivalent mass [%]	6.3	5.9	6.0

Figure 17 b. shows the first mode shape obtained from cases (a) and (b) respectively.

The second example (figures 18 a. and 18 b.) consists of a 14-story shear wall building having a setback at the fifth floor.

Geometrical and mechanical properties are:

$$E=3{,}000{,}000\ [t/m^2]$$

Shear deformations are neglected

Story	No. of 8 [m] length walls	No. of 4 [m] length walls	Thickness [m]	Weight [t]	Height [m]
14	1	4	0.25	200	3.0
13—10	1	4	0.25	400	3.0
9— 5	1	4	0.30	400	3.0
4— 1	3	4	0.40	1200	3.0

Results obtained by a discrete matrix solution (a) and transfer matrix method (b) are

Vibration parameter	Case (a)	Case (b)
1st. period [sec]	1.194	1.229
2nd. period [sec]	0.242	0.239
3rd. period [sec]	0.106	0.101
1st. equivalent mass [%]	40.0	38.2
2nd. equivalent mass [%]	26.1	21.2
3rd. equivalent mass [%]	14.1	13.7

Figure 18 c. shows the first mode shape for both solutions. They can not be differentiated in the graphical representation.

The method to analyze structures under the action of seismic loading proposed in Refer. (2), provides a fast tool to make a preliminary design. The accuracy of the results, however depend very much on the values of the stiffness parameters of the structural components.

The second example given in the text, shows a close agreement of the main vibration parameter values as compared to those obtained from a discrete matrix model. The results of the first example, however, do not have such a good agreement, showing the effect of the approximation used for the calculation of the frame stiffness parameter.

The authors are presently working on a revised version of the model to improve the estimation of the stiffness parameters and at the same time to enable the users to include some additional effects such as foundation rotation, axial and shear deformation and non linear behaviour.

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2. Guendelman, T. and Monge, J., "Preliminary Seismic Analysis of Reinforced Concrete Tall Buildings" 5WCEE, paper N°249, Rome, June 1973.
3. Muto, K., "Seismic Analysis of Reinforced Concrete Buildings," World Conference on Earthquake Engineering, Berkeley, California, June 1956.

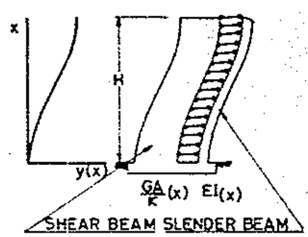


Fig. 1. Mathematical Model

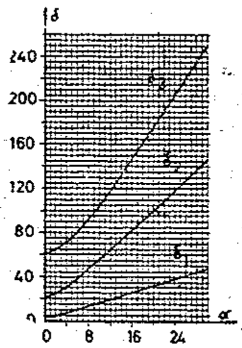
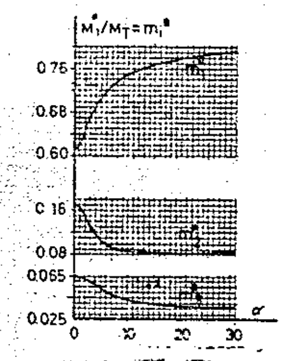


Fig. 2. Frequency Coeffcient



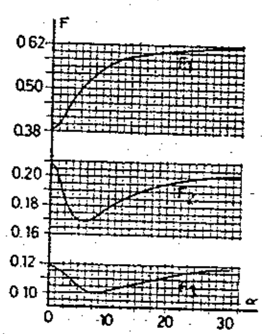


Fig. 3. Equivalent Masses m_i^*

Fig. 4. Scaling Factor F_i

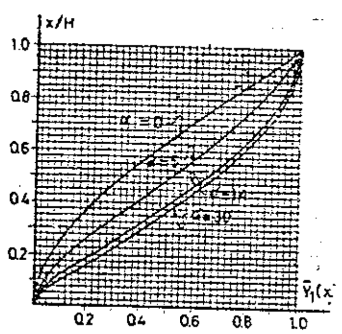


Fig. 5. First Mode Shape $\overline{Y}_1(x)$

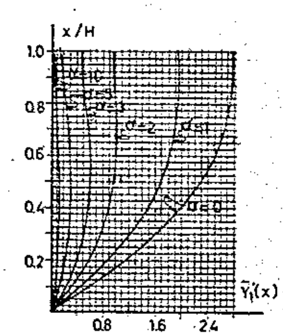


Fig. 6. First Mode Derivative $\overline{Y}'_1(x)$

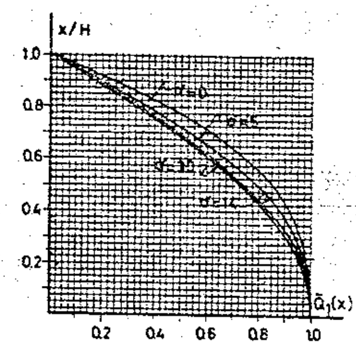


Fig. 7. First Mode Shear Dia. $\overline{Q}_1(x)$

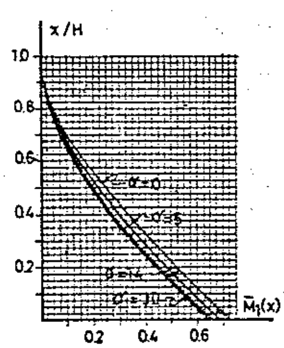


Fig. 8. First Mode Moment Dia. $\overline{M}_1(x)$

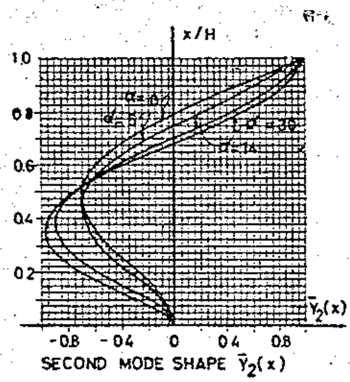


Fig. 9.

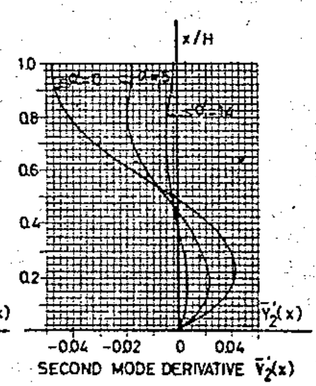


Fig. 10.

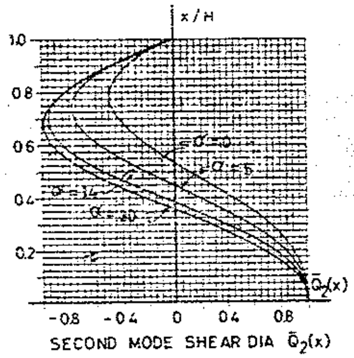


Fig. 11.

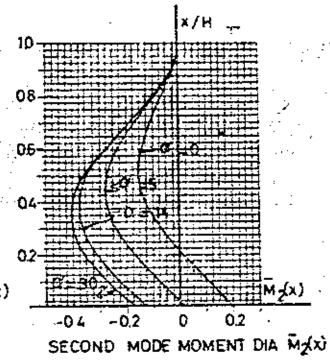


Fig. 12.

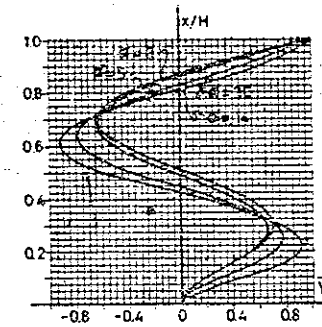


Fig. 13. Third Mode Shape $\bar{Y}_3$ (x)

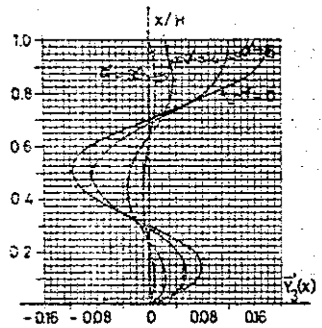


Fig. 14. Third Mode Derivative $\bar{Y}'_3$ (x)

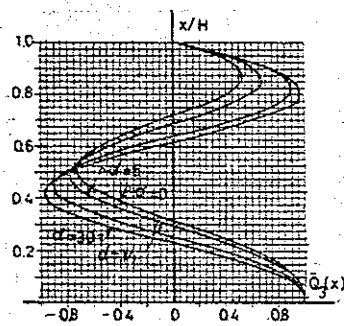


Fig. 15. Third Mode Shear Dia. $\bar{Q}_3$ (x)

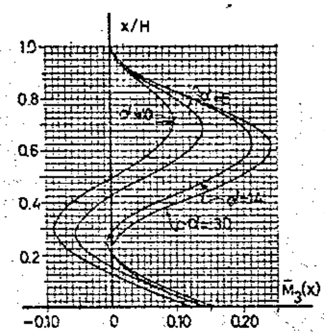


Fig. 16. Third Mode Moment Dia. $\bar{M}_3$ (x)

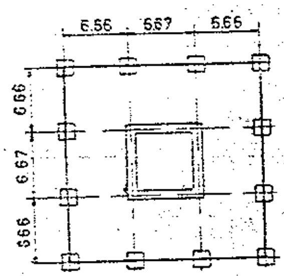


Fig. 17a. Plan of 20-Story Shear Wall-Frame Building

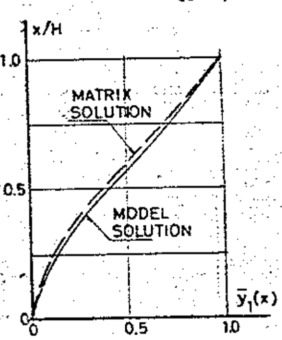


Fig. 17b. First Mode Shape

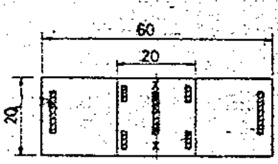


Fig. 18a. Plan of 14-Story Shear Wall Building

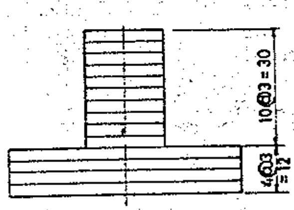


Fig. 18b. Elevation of 14-Story Shear Wall Building

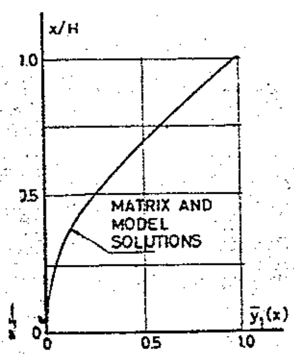


Fig. 18c. First Mode Shape